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Moran sets in complete metric space*

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Abstract The Hausdorff dimensions and packing dimensions of Moran sets in the (C,s)-homogeneous space are presented, which generalizes the results of Hua and Rao.

Keywords: (C,s)-homogeneous space, Moran set, Hausdorff dimension, packing dimension.

Supposing that (X,d) is a complete metric space, $I \subset X$ a compact set with nonempty interior in X (for convenience, we assume that the diameter of I is 1). Let $\{n_k\}_{k\geqslant 1}$ be a sequence of positive integers, and $\{c_{k,j}\}_{1\leqslant j\leqslant n_k}$ a sequence of real numbers. We denote by $\mathscr{M}(I, \{n_k\}, \{c_{k,j}\})$ the collection of Moran sets determined by I, $\{n_k\}$ and $\{c_{k,j}\}$ (for the definition of Moran set, see ref. [1]). Define $s_k(k\geqslant 1)$ by

$$\prod_{i=1}^{k}(c_{i,1}^{s_k}+\cdots+c_{i,n_i}^{s_k}) = 1.$$

and let $s_* = \liminf_{k\to\infty} s_k$, $s^* = \limsup_{k\to\infty} s_k$.

If X is a Euclidean space, and $E\in\mathscr{M}(I,\{n_k\},\{c_{k,j}\})$, under some special conditions on the sequences $\{c_{k,j}\}$, Hua and Rao separotely obtained the following results:

$$\dim E = s_*,\ \mathrm{Dim} E = s^*,$$

where $\dim E$ and $\mathrm{Dim} E$ denote the Hausdorff and packing dimensions respectively. If there is not any limitation on the sequence $\{c_{k,j}\}$, then the above results are not true (see, for example, homogeneous Cantor set and partial homogeneous Cantor set in ref. [2]).

On the other hand, if X is a general metric space, although $\{c_{k,j}\}$ satisfies the conditions given in refs. [1, 3, 4], the above results are not true either (see example 1 in ref. [5]).

In this paper, we assume that X is the (C,s)-homogeneous space (of course, Euclidean space R^n is (C_n,n)-homogeneous for some C_n. For the definition of (C,s)-homogeneous space, see ref. [6]), and give no limitation on $\{c_{k,j}\}$. We will determine the Hausdorff and packing dimensions of Moran sets in X. Our main results are as follows.

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Theorem 1. *Let X be the (C,s)-homogeneous space, then for any $E \in \mathscr{M}(I, \{n_k\}, \{c_{k,j}\})$, we have $s_* - sv \leqslant \dim E \leqslant s_*$.*

Theorem 2. *Let X be the (C,s)-homogeneous space, then for any $E \in \mathscr{M}(I, \{n_k\}, \{c_{k,j}\})$, we have $s^* \leqslant \operatorname{Dim} E \leqslant s^*(1+\eta)$, where $v = \limsup_{k\to\infty} \frac{\lg d_k}{\lg M_k}$, $\eta = \limsup_{k\to\infty} \frac{\lg d_k}{\lg M_{k-1}}$, $M_k = \max\{|I_\sigma| : \sigma \in D_k\}$ and $d_k = \min\{c_{k,j}: 1 \leqslant j \leqslant n_k\}$.*

Remark 1. By the above theorems and the conditions given in refs. [1, 3, 4], it is immediately known that the results in refs. [1, 3, 4] are also true in the case where X is the (C,s)-homogeneous space, hence these results generalize the results in refs. [1, 3, 4] from the Euclidean space to the (C,s)-homogeneous space.

Remark 2. The packing dimension of Theorem 2 is defined by the radius-based packing measure, for the definitions and properties of the radius packing measures and radius packing dimensions, one can refer to reference [7].

1 Proof of Theorem 1

To prove Theorem 1, the following lemmas will be used.

Lemma 1[6]. Let X be a complete (C,s)-homogeneous space. Then for any $t > s$, there exist a non-degenerate Borel measure μ and a constant $c \geqslant 1$ such that $0 < \mu(B(x,r)) < \infty$, and

$$\mu(B(x,\lambda r)) \leqslant c\lambda^t \mu(B(x,r)) \tag{1}$$

for any $x \in X$, $r > 0$ and $\lambda \geqslant 1$.

Let $A \subset X$. $\mathscr{H}^\alpha(A)$ denotes the α-dimensional Hausdorff measure of A. The α-dimensional Hausdorff measure of A determined by the set $\mathscr{F}$ is defined by

$$\mathscr{H}^\alpha_{\mathscr{F}}(A) = \lim_{\delta\to 0}\inf\left\{\sum |U_i|^\alpha : A \subset \bigcup_i U_i,\ U_i \in \mathscr{F} \text{ and } |U_i| < \delta\right\},$$

where $\mathscr{F}$ is defined in reference [1].

Lemma 2. *Let X be the (C,s)-homogeneous space, $E \in \mathscr{M}(I, \{n_k\}, \{c_{k,j}\})$. Then for any $\beta > v$ and $t > s$, there exists a constant $p > 0$ such that*

$$p\mathscr{H}^{\alpha+t\beta}_{\mathscr{F}}(E) \leqslant \mathscr{H}^\alpha(E) \leqslant \mathscr{H}^\alpha_{\mathscr{F}}(E).$$

Proof. The second inequality is obvious. Now, for any $\beta > \limsup_{k\to\infty}\frac{\lg d_k}{\lg M_k}$, there exists $k_0 > 0$ such that for any $k > k_0$,

$$\beta > \log_{M_k} d_k. \tag{2}$$

Let $\{U_i\}_{i\geq 1}$ be any δ-covering of E, $\sup\{|U_i|\} < \min\{|I_\sigma|: \sigma\in D_{k_0}\}$, and $U_i\cap E\neq\emptyset$, and let $b_i = |U_i|$. Since $\mathring{I}\neq\emptyset$, there exist $x\in\mathring{I}$ (open set) and $\varepsilon>0$ such that $U(x,\varepsilon)\subset\mathring{I}$, where $U(x,\varepsilon)$ denotes the open ball with center x and radius ε. For any $i\in N$, set

$$A(U_i) = \{I_\sigma: |I_\sigma|\leqslant b_i < |I_{\sigma^*}|,\ I_\sigma\cap U_i\neq\emptyset\},$$

where σ^* is obtained by deleting the last letter of σ. Put

$$D(U_i,k) = \{\sigma\in D_k: I_\sigma\in A(U_i)\},$$

$$\Delta_i = \bigcup_{k\geqslant k_0} D(U_i,\ k).$$

For the given U_i, the fact that $|I_\sigma|\to 0$ $(k\to\infty)$ implies that there exists $k_1\geqslant k_0$ such that for any $k>k_1$, $D(U_i,k)=\emptyset$; therefore the above union is a finite one. Set

$$N(U_i,k) = \# D(U_i,\ k).$$

Then $\sum_{k=k_0}^{\infty} N(U_i,\ k)$ is the finite sum, and the cardinality of Δ_i satisfies $\#\Delta_i<\infty$. By the definition of Δ_i, we know that for any τ^1 and $\tau^2\in\Delta_i$, there exists no σ such that $\tau^1=\tau^2*\sigma$, or $\tau^2=\tau^1*\sigma$; therefore the open balls $\{S_\sigma(U(x,\varepsilon))\}_{\sigma\in\Delta_i} = \{U(S_\sigma(x),\ \varepsilon c_\sigma)\}_{\sigma\in\Delta_i}$ are pairwise disjoint, where $c_\sigma = c_{1,\sigma_1}c_{2,\sigma_2}\cdots c_{k,\sigma_k}$ for $\sigma=(\sigma_1,\cdots,\sigma_k)$.

On the other hand, for any $t>s$, there exists a Borel measure μ satisfying (1) by Lemma 1. Let $\mu(B(S_\tau(x),\ \frac{1}{2}b_i\varepsilon))$ be the minimum of $\{\mu(B(S_\sigma(x),\ \frac{1}{2}b_i\varepsilon))\}_{\sigma\in\Delta_i}$. Then for any $\sigma\in\Delta_i$, we have

$$U(S_\sigma(x),\varepsilon c_\sigma)\subset U(S_\tau(x),b_i+|I_\tau|+\max_{\sigma\in\Delta_i}(|I_\sigma|+\varepsilon c_\sigma))\subset U(S_\tau(x),(3+\varepsilon)b_i). \quad (3)$$

Using eq. (1) and the definitions of $A(U_i)$ and $D(U_i,\ k)$, we have

$$\sum_{k=k_0}^{\infty}\sum_{\sigma\in D(U_i,k)}|I_\sigma|^{\beta t}\leqslant\sum_{k=k_0}^{\infty}\sum_{\sigma\in D(U_i,k)}M_k^{\beta t}<\sum_{k=k_0}^{\infty}\sum_{\sigma\in D(U_i,k)}d_k^t$$

$$<\frac{1}{\mu\left(B\left(S_\tau(x),\frac{1}{2}b_i\varepsilon\right)\right)}\sum_{k=k_0}^{\infty}\sum_{\sigma\in D(U_i,k)}d_k^t\mu\left(B\left(S_\sigma(x),\frac{1}{2}b_i\varepsilon\right)\right)$$

$$<\frac{c}{\mu\left(B\left(S_\tau(x),\frac{1}{2}b_i\varepsilon\right)\right)}\sum_{k=k_0}^{\infty}\sum_{\sigma\in D(U_i,k)}\mu\left(B\left(S_\sigma(x),\frac{1}{2}\varepsilon c_\sigma\right)\right)$$

$$< \frac{c}{\mu\left(B\left(S_\tau(x), \frac{1}{2} b_i \varepsilon\right)\right)} \mu(B(S_\tau(x), (3+\varepsilon) b_i))$$

$$< c^2(6+2\varepsilon)^t \varepsilon^{-t}. \tag{4}$$

Hence

$$\sum_{i=1}^{\infty} \sum_{\sigma \in \Delta_i} |I_\sigma|^{\alpha+\beta t} = \sum_{i=1}^{\infty} \sum_{k=k_0}^{\infty} \sum_{\sigma \in D(U_i, k)} |I_\sigma|^\alpha |I_\sigma|^{\beta t}$$

$$\leqslant \sum_{i=1}^{\infty} \sum_{k=k_0}^{\infty} \sum_{\sigma \in D(U_i, k)} |U_i|^\alpha |I_\sigma|^{\beta t}$$

$$= \sum_{i=1}^{\infty} \Big(\sum_{k=k_0}^{\infty} \sum_{\sigma \in D(U_i, k)} |I_\sigma|^{\beta t} \Big) |U_i|^\alpha$$

$$< c^2(6+2\varepsilon)^t \varepsilon^{-t} \sum_{i=1}^{\infty} |U_i|^\alpha.$$

Noticing that $\{A(U_i)\}_{i \geqslant 1} \subset \mathscr{F}$ is a δ-covering of E, and choosing $p = c^2(6+2\varepsilon)^t \varepsilon^{-t}$, we have $\mathscr{H}^\alpha(E) \geqslant p \mathscr{H}_{\mathscr{F}}^{\alpha+\beta t}(E)$. Thus, the proof of Lemma 2 is completed.

Corollary 1. *Let* $\dim_{\mathscr{F}} E$ *be the Hausdorff dimension of* E *induced by* $\mathscr{H}_{\mathscr{F}}^{\alpha}(E)$. *Then*

$$\dim_{\mathscr{F}}(E) - vs \leqslant \dim E \leqslant \dim_{\mathscr{F}} E.$$

Proof. The second part of the inequality is obvious. Now let $\dim E < \alpha$. Then $\mathscr{H}^\alpha(E) = 0$. From Lemma 2 it follows that

$$\dim_{\mathscr{F}} E \leqslant \alpha + \beta t,$$

for any $\beta > v$, and $t > s$. Since α, β and t are arbitrary, we have $\dim E \geqslant \dim_{\mathscr{F}} E - vs$. So the proof of Corollary 1 is completed.

Proof of Theorem 1. Using the net measure methods in ref. [3], we have $\dim_{\mathscr{F}} E = s_*$. By Corollary 1, the result of Theorem 1 can be immediately obtained. This completes the proof of Theorem 1.

Corollary 2. *Let* X *be the* (C, s)*-homogeneous space, and* $E \in \mathscr{M}(I, \{n_k\}, \{c_{k,j}\})$. *If* $\lim\limits_{k \to \infty} \frac{\lg d_k}{\lg M_k} = 0$, *then* $\dim E = s_*$.

2 Proof of Theorem 2

Let $A \subset X$. The upper box dimension is defined by[7]

$$\Delta(A) = \limsup_{r \to 0} \frac{\lg N_r(A)}{-\lg r},$$

where $N_r(A)$ is the minimum number of closed balls of diameter r covered A. Denote by Dim (A) the packing dimension induced by the radius-based packing measure. Then[7]

$$\mathrm{Dim}(A) = \inf\{\sup_i \Delta(A_i),\ A \subset \bigcup A_i\}.$$

Let $E \in \mathscr{M}(I, \{n_k\}, \{c_{k,j}\})$, and for any $F \subset E$, put

$$P^{\alpha}_{\mathscr{F}}(F) = \limsup_{\delta \to 0}\Big\{\sum_i |A_i|^{\alpha} : \{A_i\} \subset \mathscr{F},\ \mathring{A}_i \cap \mathring{A}_j = \emptyset,$$

$$i \neq j,\ A_i \cap \bar{F} \neq \emptyset,\ 0 < |A_i| < \delta\Big\},$$

$$\mathscr{P}^{\alpha}_{\mathscr{F}}(E) = \inf\Big\{\sum_i P^{\alpha}_{\mathscr{F}}(F_i) : E \subset \bigcup F_i\Big\},$$

$$\Delta_{\mathscr{F}}(E) = \inf\{\alpha : P^{\alpha}_{\mathscr{F}}(E) = 0\}.$$

Lemma 3. *Let X be the $(C,\ s)$-homogeneous space, $E \in \mathscr{M}(I, \{n_k\}, \{c_{k,j}\})$. Then*

$$\Delta_{\mathscr{F}}(E) \leqslant \Delta(E).$$

Proof. Similar to the proof of the lemma in ref. [4], we have

$$\Delta_{\mathscr{F}}(E) = \limsup_{r \to 0} \frac{\lg M_r}{-\lg r}, \tag{5}$$

where M_r is the maximum number of the sets $\{U\} \subset \mathscr{F}$ that are not pairwise overlap and diam$(U) \in \left(\frac{r}{2}, r\right)$.

Let N_r be the minimum number of the closed balls of diameters r covered E, and let $V_1, \cdots, V_{N_r}$ be such closed balls. Similarly, suppose that $A_1, \cdots, A_{M_r}$ are such sets $\{U\} \subset \mathscr{F}$ that are not pairwise overlap and diam $(U) \in \left(\frac{r}{2}, r\right)$. For each $A_j (j = 1, \cdots, M_r)$, there exists $\sigma = (\sigma_1, \cdots, \sigma_k) \in D_k$ such that $A_j = S_\sigma(I)$, and

$$\frac{r}{2} \leqslant c_\sigma \leqslant r.$$

Since $\mathring{I} \neq \emptyset$, there exists $x \in \mathring{I}$ and $\varepsilon > 0$ such that $U(x, \varepsilon) \subset \mathring{I}$, and $S_\sigma(U(x, \varepsilon)) = U(S_\sigma(x), \varepsilon c_\sigma) \supset U\left(S_\sigma(x), \frac{r}{2}\varepsilon\right)$. The fact that $A_j \cap E \neq \emptyset$ implies that A_j must intersect some V_k. Let

y_k be the centers of the closed balls V_k, $k = 1, \cdots, N_r$. If A_j intersects V_k, then $A_j \subset B(y_k, 2r)$, and

$$U(S_\sigma(x), \frac{\varepsilon}{2} r) \subset B(y_k, 2r).$$

Put

$$\Delta(V_k) = \{A_j : A_j \cap V_k \neq \emptyset\}.$$

If A_i, $A_j \in \Delta(V_k)$, and $i \neq j$, then there exist $\sigma \neq \tau$ such that $S_\sigma(I) = A_i$, $S_\tau(I) = A_j$. By the above argument and $\mathring{A}_i \cap \mathring{A}_j = \emptyset$, we have

$$\mathrm{dist}(S_\sigma(x), S_\tau(x)) > \frac{\varepsilon}{2} r. \tag{6}$$

Since X is (C, s)-homogeneous, the closed ball $B(y_k, 2r)$ contains at most $C\left(\frac{4}{\varepsilon}\right)^s$ points with mutual distances at least $\frac{\varepsilon}{2} r$; thus

$$\# \Delta(V_k) \leqslant C\left(\frac{4}{\varepsilon}\right)^s.$$

Therefore

$$M_r \leqslant C\left(\frac{4}{\varepsilon}\right)^s N_r. \tag{7}$$

From eqs. (5) and (7) we have

$$\Delta_{\mathscr{F}}(E) \leqslant \Delta(E).$$

Thus, the proof of Lemma 3 is completed.

Proof of Theorem 2. Let $\alpha < s^*$; then there exists the infinite number of positive integers $k \in N$, such that $\alpha < s_k$. Thus $\sum_{\sigma \in D_k} | I_\sigma |^\alpha \geqslant | I |^\alpha > 0$. By Lemma 3 we get

$$\Delta(E) \geqslant \Delta_{\mathscr{F}}(E) \geqslant s^*. \tag{8}$$

On the other hand, let $\alpha > s^*$: then there exists $k_0 > 0$ such that $s_k < \alpha$ for any $k > k_0$. For $0 < r < 1$, put $Q = \{\sigma \in D : | I_\sigma | \leqslant r < | I_{\sigma^*} |\}$. By the proof of theorem in ref. [4], we have

$$\sum_{\sigma \in Q} | I_\sigma |^\alpha \leqslant | I |^\alpha. \tag{9}$$

Let k_1 and k_2 be the maximum and minimum of the lengths of elements in Q respectively, and put $d_p = \min\{d_k : k_2 \leqslant k \leqslant k_1\}$. From (9) we have $\mathrm{Card}\, Q \leqslant (d_p r)^{-\alpha} | I |^\alpha$, and therefore

$$\begin{aligned} \Delta(E) &\leqslant \limsup_{r \to 0} \frac{\lg (d_p r)^{-\alpha} | I |^\alpha}{-\lg r} = \alpha \limsup_{r \to 0}\left(1 + \frac{\lg d_p}{\lg r}\right) \\ &\leqslant \alpha\left(1 + \limsup_{k \to 0} \frac{\lg d_k}{\lg M_{k-1}}\right). \end{aligned} \tag{10}$$

From eqs. (8) and (10),

$$s^* \leqslant \Delta(E) \leqslant s^*(1+\eta).$$

By Lemma 3.3 in ref. [8] we get

$$s^* \leqslant \mathrm{Dim}E \leqslant s^*(1+\eta).$$

By Lemma 3.3 in ref. [8] we get

$$\mathrm{Dim}(E) = \Delta(E);$$

thus

$$s^* \leqslant \mathrm{Dim}(E) \leqslant s^*(1+\eta).$$

Corollary 3. *Let X be the (C, s)-homogeneous space, and $E \in \mathscr{M}(I, \{n_k\}, \{c_{k,j}\})$. If $\lim\limits_{k\to\infty}\dfrac{\lg d_k}{\lg M_{k-1}} = 0$, then $\mathrm{Dim}E = s^*$.*

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